On Rational Homotopy and Minimal Models

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Abstract

We prove a result that enables us to calculate the rational homotopy of a wide class of spaces by the theory of minimal models. The latter are derivided from properties of the de Rham complex.

1 Differential Graded Algebras

Given a manifold M, one can consider the complex of its differential forms $(\Omega(M), d)$, which has the structure of a so-called differential graded algebra. Such differential graded algebras are the main objects of rational homotopy theory.

The idea of rational homotopy is to ignore the torsion in standard homotopy theory. Sullivan [12] showed in the 1960s that not only the simplicial homology $H_*(X, \mathbb{Z})$ and the higher homotopy groups $\pi_i(X)$, $i > 1$, of a simply-connected space X can be localised to $H_*(X, \mathbb{Q})$ and $\pi_i(X) \otimes \mathbb{Q}$. It is also possible to *geometrically* localise the space X to a space X_0 via a continuous map $X \to X_0$ which induces isomorphisms $H_*(X, \mathbb{Q}) \to H_*(X_0, \mathbb{Z})$ and $\pi_i(X) \otimes \mathbb{Q} \to \pi_i(X_0)$. The *rational homotopy type of* X is then defined as the weak homotopy type of X_0. A principal feature of rational homotopy theory, as developed by Quillen [10], is that the *geometric* localisation X_0 can be understood within an entirely *algebraic* category. This led to Sullivan's choice [13] of a particular algebraic category that models exactly the rational homotopy type of a space. It is to this category – the category of minimal differential graded algebras – that we turn now.

Let $\mathbb{K}$ be a field of characteristic zero. A *differential graded algebra (DGA)* is a graded $\mathbb{K}$-algebra $A = \bigoplus_{i \in \mathbb{N}} A^i$ together with a $\mathbb{K}$-linear map $d\colon A \to A$ such that $d(A^i) \subset A^{i+1}$ and the following conditions are satisfied:

(i) The $\mathbb{K}$-algebra structure of A is given by an inclusion $\mathbb{K} \hookrightarrow A^0$.

(ii) The multiplication is graded commutative, i.e. for $a \in A^i$ and $b \in A^j$ one has $a \cdot b = (-1)^{i \cdot j} b \cdot a \in A^{i+j}$.

(iii) The Leibniz rule holds: $\forall_{a \in A^i} \forall_{b \in A}\ d(a \cdot b) = d(a) \cdot b + (-1)^i a \cdot d(b)$

(iv) The map d is a differential, i.e. $d^2 = 0$.

Further, we define $|a| := i$ for $a \in A^i$.

The *i-th cohomology of a DGA* (A, d) is the algebra

$$H^i(A, d) := \frac{\ker(d\colon\ A^i \to A^{i+1})}{\mathrm{im}(d\colon\ A^{i-1} \to A^i)}.$$

If (B, d_B) is another DGA, then a $\mathbb{K}$-linear map $f\colon\ A \to B$ is called *morphism* if $f(A^i) \subset B^i$, f is multiplicative, and $d_B \circ f = f \circ d_A$. Obviously, any such f induces a homomorphism $f^*\colon\ H^*(A, d_A) \to H^*(B, d_B)$. A morphism of differential graded algebras inducing an isomorphism on cohomology is called *quasi-isomorphism*.

Definition 1.1. A DGA $(\mathcal{M}, d)$ is said to be *minimal* if

(i) there is a graded vector space $V = \left(\bigoplus_{i\in\mathbb{N}_+} V^i\right) = \mathrm{Span}\,\{a_k \mid k \in I\}$ with homogeneous elements a_k, which we call the generators,

(ii) $\mathcal{M} = \bigwedge V$,

(iii) the index set I is well ordered, such that $k < l \ \Rightarrow |a_k| \leq |a_l|$ and the expression for da_k contains only generators a_l with $l < k$.

We shall say that $(\mathcal{M}, d)$ is a *minimal model for a differential graded algebra* (A, d_A) if $(\mathcal{M}, d)$ is minimal and there is a quasi-isomorphism of DGAs $\rho\colon\ (\mathcal{M}, d) \to (A, d_A)$, i.e. ρ induces an isomorphism $\rho^*\colon\ H^*(\mathcal{M}, d) \to H^*(A, d_A)$ on cohomology.

The importance of minimal models is reflected by the following theorem, which is taken from Sullivan's work [13, Section 5].

Theorem 1.2. *A differential graded algebra* (A, d_A) *with* $H^0(A, d_A) = \mathbb{K}$ *possesses a minimal model. It is unique up to isomorphism of differential graded algebras.*

We quote the existence-part of Sullivan's proof, which gives an explicit construction of the minimal model. Whenever we are going to construct such a model for a given algebra in this note, we will do it as we do it in this proof.

Proof of the existence. We need the following algebraic operations to "add" resp. "kill" cohomology.

Let $(\mathcal{M}, d)$ be a DGA. We "add" cohomology by choosing a new generator x and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(x), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(x) = 0,$$

and "kill" a cohomology class $[z] \in H^k(\mathcal{M}, d)$ by choosing a new generator y of degree $k - 1$ and setting

$$\widetilde{\mathcal{M}} := \mathcal{M} \otimes \bigwedge(y), \quad \tilde{d}|_{\mathcal{M}} = d, \quad \tilde{d}(y) = z.$$

Note that z is a polynomial in the generators of $\mathcal{M}$.

Now, let (A, d_A) a DGA with $H^0(A, d_A) = \mathbb{K}$. We set $\mathcal{M}_0 := \mathbb{K}$, $d_0 := 0$ and $\rho_0(x) = x$.

Suppose now $\rho_k\colon (\mathcal{M}_k, d_k) \to (A, d_A)$ has been constructed so that ρ_k induces isomorphisms on cohomology in degrees $\leq k$ and a monomorphism in degree $(k+1)$.

"Add" cohomology in degree $(k+1)$ to get a morphism of differential graded algebras $\rho_{(k+1),0}\colon (\mathcal{M}_{(k+1),0}, d_{(k+1),0}) \to (A, d_A)$ which induces an isomorphism $\rho^*_{(k+1),0}$ on cohomology in degrees $\leq (k+1)$. Now, we want to make the induced map $\rho^*_{(k+1),0}$ injective on cohomology in degree $(k+2)$.

We "kill" the kernel on cohomology in degree $(k+2)$ (by non-closed generators of degree (k+1)) and define $\rho_{(k+1),1}\colon (\mathcal{M}_{(k+1),1}, d_{(k+1),1}) \to (A, d_A)$ accordingly. If there are generators of degree one in $(\mathcal{M}_{(k+1),0}, d_{(k+1),0})$ it is possible that this killing process generates new kernel on cohomology in degree $(k+2)$. Therefore, we may have to "kill" the kernel in degree $(k+2)$ repeatedly.

We end up with a morphism $\rho_{(k+1),\infty}\colon (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty}) \to (A, d_A)$ which induces isomorphisms on cohomology in degrees $\leq (k+1)$ and a monomorphism in degree $(k+2)$. Now, we are going to set $\rho_{k+1} := \rho_{(k+1),\infty}$ and $(\mathcal{M}_{k+1}, d_{k+1}) := (\mathcal{M}_{(k+1),\infty}, d_{(k+1),\infty})$.

Inductively we get the minimal model $\rho\colon (\mathcal{M}, d) \to (A, d_A)$. □

A *minimal model* $(\mathcal{M}_M, d)$ *of a connected smooth manifold* M is a minimal model for the de Rahm complex $(\Omega(M), d)$ of differential forms on M. Note that this implies that $(\mathcal{M}, d)$ is an algebra over $\mathbb{R}$. The last theorem implies that every connected smooth manifold possesses a minimal model which is unique up to isomorphism of differential graded algebras.

For a certain class of spaces that includes all nilpotent (and hence all simply-connected) spaces, we can read off the non-torsion part of the homotopy from the generators of the minimal model. (The definition of a nilpotent space will be given below.)

2 Rational Minimal Models

In general, it is very difficult to calculate the homotopy groups $\pi_k(X)$ of a given topological space X. However, if one is willing to forget the torsion, with certain assumptions on X, the rational homotopy groups $\pi_k(X) \otimes \mathbb{Q}$ can be determined by the theory of minimal models.

In order to relate minimal models to rational homotopy theory, we need a differential graded algebra over $\mathbb{Q}$ to replace the de Rahm algebra.

Let Δ^n be a standard simplex in $\mathbb{R}^{n+1}$ and $(\Omega_{PL}(\Delta^n), d)$ the restriction to Δ^n of all differential forms in $\mathbb{R}^{n+1}$ which can be written as $\sum P_{i_1 \ldots i_k} dx_{i_1} \ldots dx_{i_k}$, where $P_{i_1 \ldots i_k} \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$ together with multiplication and differential induced by $\mathbb{R}^{n+1}$.

Let $X = \{(\sigma_i)_{i \in I}\}$ be a path-connected simplicial complex. Set for $k \in \mathbb{Z}$

$$\Omega^k_{PL}(X) := \{(\alpha_i)_{i \in I} \mid \alpha_i \in \Omega^k_{PL}(\sigma_i) \wedge (\sigma_i \subset \partial\sigma_j \Rightarrow \alpha_j|_{\sigma_i} = \alpha_i)\},$$

and $\Omega_{PL}(X) := \bigoplus_{k \in \mathbb{Z}} \Omega^k_{PL}(X)$. It can be verified that the set $\Omega_{PL}(X)$ of so-called *PL forms* is a differential graded algebra over $\mathbb{Q}$ if we use the multiplication and the differential on forms componentwise.

Analogous to the usual result for the de Rham complex, we have:

Theorem 2.1 ([9, Theorem 1.1.4])**.** *If X is a path-connected simplicial complex, then there is an isomorphism $H^*(\Omega_{PL}(X), d) \cong H^*(X, \mathbb{Q})$.* □

For such a simplicial complex X, we define the *($\mathbb{Q}$-)minimal model* $\mathcal{M}_{X,\mathbb{Q}}$ of X to be the minimal model of $(\Omega_{PL}(X), d)$. Its relation to the minimal model of a smooth manifold is given by the following theorem.

Theorem 2.2 ([9, Theorem 1.3.9])**.** *Let M be a connected smooth manifold. Then there is an isomorphism $\mathcal{M}_{M,\mathbb{Q}} \otimes \mathbb{R} \cong \mathcal{M}_M$.* □

3 Nilpotent spaces

Already in is paper [13], Sullivan shows that for nilpotent spaces, there is a correspondence between the minimal model and the rational homotopy. To state this result, we need the notion of a nilpotent space resp. nilpotent module.

Let G be a group, H be a G-module, $\Gamma_G^0 H := H$ and

$$\Gamma_G^{i+1} H := \langle g.h - h \,|\, g \in G \wedge h \in \Gamma_G^i H \rangle \subset \Gamma_G^i H$$

for $i \in \mathbb{N}$.

Then, H is called a *nilpotent module* if there is $n_0 \in \mathbb{N}$ such that $\Gamma_G^{n_0} H = \{1\}$.

We recall the natural π_1-module structure of the higher homotopy groups π_n of a topological space. For instance, let (X, x_0) be a pointed space with universal cover $(\widetilde{X}, \widetilde{x}_0)$. It is well known that $\pi_1(X, x_0) \cong D(\widetilde{X})$, the group of deck transformations of the universal covering. Now, because $\widetilde{X}$ is simply-connected, every free homotopy class of self-maps of $\widetilde{X}$ determines uniquely a class of basepoint preserving self-maps of $\widetilde{X}$ (see e.g. [5, Proposition 4.A.2]). This means that to every homotopy class of deck transformations corresponds a homotopy class of basepoint preserving self-maps (which are, in fact, homotopy equivalences) $(\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{x}_0)$. These maps provide induced automorphisms of homotopy groups $\pi_n(\widetilde{X}, \widetilde{x}_0) \cong \pi_n(X, x_0)$ $(n > 1)$ and this whole process then provides an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Definition 3.1. A path-connected topological space X whose universal covering exists is called *nilpotent* if for $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is a nilpotent group and the higher homotopy groups $\pi_n(X, x_0)$ are nilpotent $\pi_1(X, x_0)$-modules for all $n \in \mathbb{N}$, $n \geq 2$. Note, the definition is independent of the choice of the base point.

Example.

(i) Simply-connected spaces are nilpotent.

(ii) S^1 is nilpotent.

(iii) The cartesian product of two nilpotent spaces is nilpotent. Therefore, all tori are nilpotent.

(iv) The Klein bottle is not nilpotent.

(v) $P^n(\mathbb{R})$ is nilpotent if and only if $n \equiv 1(2)$.

Proof. (i) - (iv) are obvious and (v) can be found in Hilton's book [7] on page 165. □

The main theorem on the rational homotopy of nilpotent spaces is the following.

Theorem 3.2. *Let X be a path-connected nilpotent CW-complex with finitely generated homotopy groups. If $\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V$ denotes the minimal model, then for all $k \in \mathbb{N}$ with $k \geq 2$ holds:*

$$\mathrm{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}) \cong V^k$$

Using another approach to minimal models (via localisation of spaces and Postnikow towers), this theorem is proved for example in [8]. The proof that we shall give here is new to the author's knowledge. We will show the following more general result mentioned (but not proved) by Halperin in [4].

Theorem 3.3. *Let X be a path-connected triangulable topological space whose universal covering exists. Denote by $\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V$ the minimal model and assume that*

(i) each $\pi_k(X)$ is a finitely generated nilpotent $\pi_1(X)$-module for $k \geq 2$ and

(ii) the minimal model for $K(\pi_1(X), 1)$ has no generators in degrees greater than one.

Then for each $k \geq 2$ there is an isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}) \cong V^k$.

Remark. The homotopy groups of a compact nilpotent smooth manifold are finitely generated:

By [7, Satz 7.22], a nilpotent space has finitely generated homotopy if and only if it has finitely generated homology with $\mathbb{Z}$-coefficients. The latter is satisfied for compact spaces. □

The main tool for the proof of the above theorems is a consequence of the fundamental theorem of Halperin [4]. In the next section, we quote it and use it to prove Theorems 3.2 and 3.3.

4 The Halperin-Grivel-Thomas theorem

To state the theorem, let us recall a basic construction for fibrations.

Let $\pi\colon E \to B$ be a fibration with path-connected basis B. Therefore, all fibers $F_b = \pi^{-1}(\{b\})$ are homotopy equivalent to a fixed fiber F since each path γ in B lifts to a homotopy equivalence $L_\gamma\colon F_{\gamma(0)} \to F_{\gamma(1)}$ between the fibers over the endpoints of γ. In particular, restricting the paths to loops at a basepoint of B we obtain homotopy equivalences $L_\gamma\colon F \to F$ for F the fibre over the basepoint b_0. One can show that this induces a natural $\pi_1(B, b_0)$-module structure on $H^*(F, \mathbb{Q})$.

Theorem 4.1 ([9, Theorem 1.4.4]). *Let F, E, B be path-connected triangulable topological spaces and $F \to E \to B$ a fibration such that $H^n(F, \mathbb{Q})$ is a nilpotent $\pi_1(B, b_0)$-module for $n \in \mathbb{N}_+$. The fibration induces a sequence*

$$(\Omega_{PL}(B), d_B) \longrightarrow (\Omega_{PL}(E), d_E) \longrightarrow (\Omega_{PL}(F), d_F)$$

of differential graded algebras. Suppose that $H^*(F, \mathbb{Q})$ or $H^*(B, \mathbb{Q})$ is of finite type.

Then there is a quasi-isomorphism $\Psi\colon (\mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}\,,\, D) \to (\Omega_{PL}(E), d_E)$ making the following diagram commutative:

$$\begin{array}{ccccc}
(\Omega_{PL}(B), d_B) & \longrightarrow & (\Omega_{PL}(E), d_E) & \longrightarrow & (\Omega_{PL}(F), d_F) \\
\uparrow \rho_B & & \uparrow \Psi & & \uparrow \rho_F \\
(\mathcal{M}_{B,\mathbb{Q}}\,,\, D_B) & \hookrightarrow & (\mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}\,,\, D) & \longrightarrow & (\mathcal{M}_{F,\mathbb{Q}}\,,\, D_F)
\end{array}$$

Furthermore, the left and the right vertical arrows are the minimal models. Moreover, if $\mathcal{M}_F = \bigwedge V_F$, there is an ordered basis $\{v_i^F \mid i \in I\}$ of V_F such that for all $i, j \in I$ holds $D(v_i^F) \in \mathcal{M}_B \otimes (\mathcal{M}_F)_{<v_i^F}$ and $(v_i^F < v_j^F \Rightarrow |v_i^F| \leq |v_j^F|)$. □

Remark. In general, $(\mathcal{M}_{B,\mathbb{Q}} \otimes \mathcal{M}_{F,\mathbb{Q}}\,,\, D)$ is not a minimal differential graded algebra and $D|_{\mathcal{M}_{F,\mathbb{Q}}} \neq D_F$ is possible.

We need some further preparations for the proofs of the above theorems. The first is a reformulation of the results 3.8 – 3.10 in [7]. It justifies the statement of the next theorem.

Proposition 4.2. *Let G be a finitely generated nilpotent group. Then the set $T(G)$ of torsion elements of G is a finite normal subgroup of G and $G/T(G)$ is finitely generated.* □

Theorem 4.3. *Let G be a finite generated nilpotent group and denote by $T(G)$ its finite normal torsion group.*

Then $K(G, 1)$ and $K(G/T(G), 1)$ share their minimal model.

Proof. Since $T(G)$ is finite and $\mathbb{Q}$ is a field, we get from [2, Section 4.2] $H^n(K(T(G), 1)), \mathbb{Q}) = \{0\}$ for $n \in \mathbb{N}_+$. The construction of the minimal model in the proof of Theorem 1.2 implies that $\mathcal{M}_{K(T(G),1),\mathbb{Q}}$ has no generators of degree greater than zero. Now, the theorem follows from the preceding one, applied to the fibration $K(T(G), 1) \to K(G, 1) \to K(G/T(G), 1)$. □

Lemma 4.4. *Let X be topological space with universal covering $\mathfrak{p}\colon \widetilde{X} \to X$.*

Then, up to weak homotopy equivalence of the total space, there is a fibration $\widetilde{X} \to X \to K := K(\pi_1(X), 1)$. Moreover, for a class $[\gamma] \in \pi_1(K) \cong \pi_1(X)$ the homotopy equivalences $L_{[\gamma]}\colon \widetilde{X} \to \widetilde{X}$ described at the beginning of this section are given by the corresponding deck transformations of $\mathfrak{p}$.

Proof. Denote by $\pi\colon E \to K(\pi_1(X), 1)$ the universal principal $\pi_1(X)$-bundle. Regard on $E \times \widetilde{X}$ the diagonal $\pi_1(X)$-action. Then, the fibre bundle

$$\widetilde{X} \longrightarrow \big((E \times \widetilde{X})/\pi_1(X)\big) \longrightarrow K$$

has the desired properties. □

4.1 Proof of Theorem 3.3:

Let X be as in the statement of the theorem. For simply-connected spaces, the theorem was proven in [3, Theorem 15.11]. Now, the idea is to use this result and to consider the universal cover $\mathfrak{p}\colon \widetilde{X} \to X$. Denote by $\mathcal{M}_{\widetilde{X},\mathbb{Q}} = \bigwedge \widetilde{V}$ and $\mathcal{M}_{X,\mathbb{Q}} = \bigwedge V$ the minimal models. We shall show

$$\forall_{k\geq 2}\ V^k \cong \widetilde{V}^k. \tag{1}$$

This and the truth of the theorem for simply-connected spaces implies then the general case

$$\forall_{k\geq 2}\ V^k \cong \widetilde{V}^k \cong \mathrm{Hom}_{\mathbb{Z}}(\pi_k(\widetilde{X}),\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}}(\pi_k(X),\mathbb{Q}).$$

It remains to show (1): Since X is triangulable, X and $\widetilde{X}$ can be seen as CW-complexes. Therefore, up to weak homotopy, there is the following fibration of CW-complexes

$$\widetilde{X} \longrightarrow X \xrightarrow{\ \pi\ } K(\pi_1(X),1) =: K.$$

We prove below:

$$H^*(\widetilde{X},\mathbb{Q}) \text{ is of finite type.} \tag{2}$$

$$H^*(\widetilde{X},\mathbb{Q}) \text{ is a nilpotent } \pi_1(X)\text{-module.} \tag{3}$$

Then Theorem 4.1 implies the existence of a quasi-isomorphism ρ such that the following diagram commutes:

$$\begin{array}{ccccc}
(\Omega_{PL}(K), d_K) & \longrightarrow & (\Omega_{PL}(X), d_X) & \longrightarrow & (\Omega_{PL}(\widetilde{X}), d_{\widetilde{X}}) \\
\uparrow{\scriptstyle \rho_K} & & \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho_{\widetilde{X}}} \\
(\mathcal{M}_{K,\mathbb{Q}}\,, D_K) & \hookrightarrow & (\mathcal{M}_{K,\mathbb{Q}} \otimes \mathcal{M}_{\widetilde{X},\mathbb{Q}}\,, D) & \longrightarrow & (\mathcal{M}_{\widetilde{X},\mathbb{Q}}\,, D_{\widetilde{X}})
\end{array}$$

Finally, we shall see

$$(\mathcal{M}_{K,\mathbb{Q}} \otimes \mathcal{M}_{\widetilde{X},\mathbb{Q}}\,, D) \text{ is a minimal differential graded algebra} \tag{4}$$

and this implies (1) since $\mathcal{M}_K$ has no generators of degree greater than one by assumption (ii).

We still have to prove (2) - (4):

By assumption (i), $\pi_k(X) = \pi_k(\widetilde{X})$ is finitely generated for $k \geq 2$. Since simply-connected spaces are nilpotent, [7, Satz 7.22] implies the finite generation of $H_*(\widetilde{X},\mathbb{Z})$ and (2) follows.

(3) is the statement of Theorem 2.1 $(i) \Rightarrow (ii)$ in [6] – applied to the action of $\pi_1(X)$ on $\pi_i(\widetilde{X})$.

ad (4): By assumption (ii), $\mathcal{M}_K$ has no generators in degrees greater than one, i.e. $\mathcal{M}_{K,\mathbb{Q}} = \bigwedge\{v_i \mid i \in I\}$ with $|v_i| = 1$. The construction of the minimal model in the proof of Theorem 1.2 implies that the minimal model of a simply-connected space has no generators in degree one, i.e. $\mathcal{M}_{\widetilde{X},\mathbb{Q}} = \bigwedge\{w_j \mid j \in J\}$

with $|w_j| > 1$. We expand the well orderings of I and J to a well ordering of their union by $\forall_{i \in I} \forall j \in J \ \ i < j$. Theorem 4.1 implies that $D(w_j)$ contains only generators which are ordered before w_j. Trivially, $D(v_i)$ also has this property, so we have shown (4) and the theorem is proved. $\square$

4.2 Proof of Theorem 3.2:

Let X be a path-connected nilpotent CW-complex with finitely generated fundamental group and finitely generated homotopy. By Theorem 3.3, we have to show that the minimal model of $K(\pi_1(X), 1)$ has no generators in degrees greater than one.

Theorem 4.3 implies that it suffices to show that $K(\pi_1(X)/T, 1)$ has this property, where T denotes the torsion group of $\pi_1(X)$. $\Gamma := \pi_1(X)/T$ is a finitely generated nilpotent group without torsion. By [11, Theorem 2.18], Γ can be embedded as a lattice in a connected and simply-connected nilpotent Lie group G. Therefore, the nilmanifold G/Γ is a $K(\Gamma, 1)$ and from [1, Theorem 3.11] follows that its minimal model has no generators in degrees greater than one. $\square$

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